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FREE VIBRATION ANALYSIS OF DOUBLY CURVED SHALLOW SHELLS ON RECTANGULAR PLANFORM USING THREE-DIMENSIONAL ELASTICITY THEORY

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Abstract—The free vibration analysis of homogeneous and laminated doubly curved shells on rectangular planform and made of an orthotropic material has been presented using the three-dimensional elasticity equations. A solution is obtained utilizing the asumption that the ratio of the shell thickness to its middle surface radius is negligible, as compared to unity. However, it is shown that by dividing the shell thickness into layers of smaller thickness and matching the interface displacement and stress continuity conditions, very accurate results can be obtained even for very thick shells. The two-dimensional shell theories have been compared for their accuracy in the light of the present three-dimensional elasticity analysis.

INTRODUCTION

Structural applications of laminated composite shells are ever on the increase due to their high-modulus and low-density material properties. As a consequence, analysis of such structures is gaining considerable importance over the years and demands accurate analysis. It is well known that the composite materials are anisotropic in nature and are more often treated as orthotropic materials. The analysis of such structures using the three-dimensional (3-D) elasticity theory involves considerable mathematical manipulation, as may be seen in the work of Shrinivas and Rao (1970) for the case of rectangular plates. Due to the presence of curvature, as in the case of cylindrical and spherical shells, the problem becomes unapproachable through the 3-D equations since the governing differential equations for such shells involve variable coefficients.

In spite of the presence of inherent mathematical complexities, many useful solutions to free vibration problems of cylindrical and spherical shells made of isotropic material have been given by Greenspon (1958), Gazis (1959), Shah *et al.* (1969a,b) and Eringen and Suhubi (1975). It may be noted from these studies that for shells made of isotropic materials, governing differential equations of the 3-D elasticity can be easily solved in terms of displacement potentials in which variation of displacements in the normal direction is finally expressed in terms of Bessel functions. However, a similar approach is not possible if the material is orthotropic and hence the solution is usually obtained using the Frobenius method—normally employed when solving differential equations with variable coefficients. This approach is used in the works of Chou and Achenbach (1972), Armenákas and Reitz (1973) and Srinivas (1974) for the case of closed cylindrical shells. Numerical methods such as the extended Ritz method, have been used by Nelson *et al.* (1971) to obtain the vibration frequencies of closed finite length cylinders, and by Nelson (1973) for spherical shells.

It may be observed that the success of the 3-D analysis of shells depends on the ability to solve the resulting differential equations with variable coefficients. In order to avoid going through the complex mathematical manipulations, researchers over the years have reduced the 3-D shell problem to a two-dimensional problem, as may be found in the monumental works of Flügge (1962), Gol'denveizer (1961), Sanders (1959) and Timoshenko and Woinowsky-Krieger (1959). Thus, within the framework of different shell theories the vibration problems of shells have received considerable attention, for example, see Leissa (1973), Niordson (1985) and Seide (1975).

It may be said here that, besides being 2-D approximations, most of these 2-D theories fail to satisfy the interface transverse stress continuity conditions in the case of laminated structures. This is so because firstly, most 2-D theories do not account for the transverse normal stress, and secondly since they are based on an assumed displacement form, it is very difficult to select a displacement form which results in the continuity of transverse stresses across the interfaces of an arbitrarily laminated shell. It is easy to satisfy the interface continuity conditions if the laminated shell is treated as a 3-D problem since all the stresses are now functions of the normal coordinate (z). The complex mathematical manipulations can still be avoided for some doubly curved, simply supported shells without reducing them to 2-D cases, but by reducing the governing equations to those with constant coefficients and thus retaining the 3-D characteristic of the problem. The results from such an analysis are useful in validating the less approximate 2-D theories.

It may be said here that, to date, the vibration problems of open cylindrical shell and doubly curved shell have not been solved using the analytical approach to treat the 3-D elasticity equations. In this paper an attempt has been made to solve these problems using the 3-D equations. In the present 3-D analysis of simply supported, doubly curved shallow shells of rectangular planform, the displacements are chosen to vary trignometrically in the x- and y-directions (which are the Cartesian coordinates of the projection of the middle surface on the x–y plane). The three governing coupled partial differential equations (PDEs) are reduced to three coupled ODEs with the normal coordinate (z) as the independent variable. These three coupled ODEs are then solved to obtain the complete solution. In the case of laminated shells each ply is treated as a homogeneous shell and by satisfying the interface and exterior surface conditions, a frequency determinant is set up and solved.

BASIC EQUATIONS OF THE THREE-DIMENSIONAL ELASTICITY

For an open shallow shell, the middle surface can be defined by a set of Cartesian coordinate systems as shown in Fig. 1. In the present analysis we restrict our attention to the analysis of doubly curved, shallow shells on rectangular planform with zero twist. Such surfaces are defined by $z = x^2/2R_1 + y^2/2R_2$. The paraboloid of revolution on square planform, the translational shell on rectangular planform and the spherical shell on square planform are the class of surfaces which can be treated by the present analysis. Assuming the twist of the surface to be zero, the strain displacement relations of the 3-D elasticity

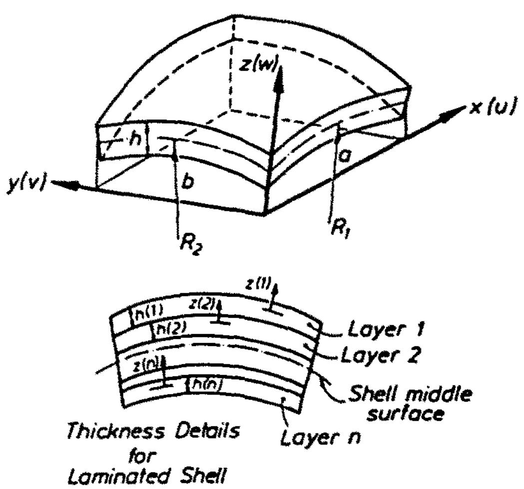

Fig. 1. The dimensions and coordinate system for a doubly curved shallow shell.

equations corresponding to the present problem are written (Saada, 1974) as

$$\varepsilon_x = \left[\frac{R_1}{R_1+z}\right]\left[\frac{\partial u}{\partial x}+\frac{w}{R_1}\right];\quad \varepsilon_y = \left[\frac{R_2}{R_2+z}\right]\left[\frac{\partial v}{\partial y}+\frac{w}{R_2}\right];\quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \left[\frac{R_1}{R_1+z}\right]\frac{\partial u}{\partial y}+\left[\frac{R_2}{R_2+z}\right]\frac{\partial v}{\partial x};\quad \gamma_{xz} = \left[\frac{R_1}{R_1+z}\right]\left[\frac{\partial w}{\partial x}-\frac{u}{R_1}\right]+\frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \left[\frac{R_2}{R_2+z}\right]\left[\frac{\partial w}{\partial y}-\frac{v}{R_2}\right]+\frac{\partial v}{\partial z}. \tag{1}$$

In these equations, x, y and z are the Cartesian coordinates in which z is measured from the middle surface of the shell; u, v and w are the displacements in the x-, y- and z-directions; R_1 and R_2 are the middle surface radii of curvature (which are assumed to be constant); ε_x, ε_y and ε_z are the normal strains in the x-, y- and z-directions; and γ_{xy}, γ_{xz} and γ_{yz} are the shear strains. In order to reduce the system of equations with variable coefficients to those with constant coefficients, the following assumptions are made:

$$\left[\frac{R_1}{R_1+z}\right]\approx 1;\quad \left[\frac{R_2}{R_2+z}\right]\approx 1. \tag{2}$$

For these assumptions to be true, it is essential that (h/R_1) and $(h/R_2) \ll 1$. Thus, eqns (2) can easily be satisfied if the shell is slightly curved or if the thickness of the shell is very small compared to the radii of curvatures. Thus in the case of thick shells, the thickness of the shell is to be divided into a number of layers with smaller thickness so that eqns (2) are satisfied. This will allow us to obtain the exact values for thick shells. Utilizing eqns (2) in eqns (1) the strain-displacement relations are rewritten as

$$\varepsilon_x = \frac{\partial u}{\partial x}+\frac{w}{R_1};\quad \varepsilon_y = \frac{\partial v}{\partial y}+\frac{w}{R_2};\quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y};\quad \gamma_{xz} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}-\frac{u}{R_1};\quad \gamma_{yz} = \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}-\frac{v}{R_2}. \tag{3}$$

The stress–strain relations for an orthotropic material read

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{12} & C_{22} & C_{23}\\ C_{13} & C_{23} & C_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{bmatrix} \tag{4a}$$

$$\tau_{xy} = C_{66}\gamma_{xy};\quad \tau_{xz} = C_{44}\gamma_{xz};\quad \tau_{yz} = C_{55}\gamma_{yz}. \tag{4b}$$

Here, σ_x, σ_y and σ_z are the normal stresses; τ_{xy}, τ_{xz} and τ_{yz} are the shear stresses; and C_{ij} are the elastic constants of the orthotropic material. Using eqn (2), the 3-D stress equilibrium equations (Saada, 1974) can be written as

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+\left[\frac{2}{R_1}+\frac{1}{R_2}\right]\tau_{xz} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+\left[\frac{1}{R_1}+\frac{2}{R_2}\right]\tau_{yz} = \rho\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+\left[\frac{1}{R_1}+\frac{1}{R_2}\right]\sigma_z-\frac{\sigma_x}{R_1}-\frac{\sigma_y}{R_2} = \rho\frac{\partial^2 w}{\partial t^2}. \tag{5}$$

In the above equations, t is the time coordinate and ρ is the density of the material per unit volume. Substituting the stress–strain relations (4), via the strain–displacement relations (3), the above stress equilibrium equations can be written in terms of three displacements as

$$C_{11}\frac{\partial^2 u}{\partial x^2}+C_{66}\frac{\partial^2 u}{\partial y^2}+C_{44}\frac{\partial^2 u}{\partial z^2}+\left[\frac{C_{44}}{R_1}+\frac{C_{44}}{R_2}\right]\frac{\partial u}{\partial z}-\left[\frac{2}{R_1}+\frac{1}{R_2}\right]\frac{C_{44}}{R_1}u$$
$$+[C_{66}+C_{12}]\frac{\partial^2 v}{\partial x\partial y}+\left[\frac{C_{11}+2C_{44}}{R_1}+\frac{C_{12}+C_{44}}{R_2}\right]\frac{\partial w}{\partial x}+[C_{13}+C_{44}]\frac{\partial^2 w}{\partial x\partial z}=\rho\frac{\partial^2 u}{\partial t^2}.$$

$$C_{66}\frac{\partial^2 v}{\partial x^2}+C_{22}\frac{\partial^2 v}{\partial y^2}+C_{55}\frac{\partial^2 v}{\partial z^2}+\left[\frac{C_{55}}{R_1}+\frac{C_{55}}{R_2}\right]\frac{\partial v}{\partial z}-\left[\frac{1}{R_1}+\frac{2}{R_2}\right]\frac{C_{55}}{R_2}v$$
$$+[C_{66}+C_{12}]\frac{\partial^2 u}{\partial x\partial y}+\left[\frac{C_{22}+2C_{55}}{R_2}+\frac{C_{12}+C_{55}}{R_1}\right]\frac{\partial w}{\partial y}+[C_{23}+C_{55}]\frac{\partial^2 w}{\partial y\partial z}=\rho\frac{\partial^2 v}{\partial t^2}.$$

$$\left[\frac{C_{13}-C_{11}-C_{44}}{R_1}+\frac{C_{13}-C_{12}}{R_2}\right]\frac{\partial u}{\partial x}+[C_{13}+C_{44}]\frac{\partial^2 u}{\partial x\partial z}+[C_{23}+C_{55}]\frac{\partial^2 v}{\partial y\partial z}$$
$$+\left[\frac{C_{23}-C_{22}-C_{55}}{R_2}+\frac{C_{23}-C_{12}}{R_1}\right]\frac{\partial v}{\partial y}+C_{44}\frac{\partial^2 w}{\partial x^2}+C_{55}\frac{\partial^2 w}{\partial y^2}+C_{33}\frac{\partial^2 w}{\partial z^2}$$
$$+\left[\frac{1}{R_1}+\frac{1}{R_2}\right]C_{33}\frac{\partial w}{\partial z}+\left[\frac{C_{13}-C_{11}}{R_1^2}+\frac{C_{13}+C_{23}-2C_{12}}{R_1R_2}+\frac{C_{23}-C_{22}}{R_2^2}\right]w=\rho\frac{\partial^2 w}{\partial t^2}. \quad (6)$$

The above equations are the required equilibrium equations and they are differential equations with constant coefficients. Had we not made use of the assumptions in eqns (2) the above equilibrium equations would have had coefficients involving the normal coordinate z, the solution of which would have caused a great deal of mathematical difficulties. In the next section, the solution of eqns (6) is sought using the method of separation of variables.

SOLUTION OF THE 3-D EQUILIBRIUM EQUATIONS

The solution of the equilibrium eqns (6) is difficult to obtain for a given general boundary and surface conditions. However, all-round simply supported shells render the solution of these equations in terms of trigonometric functions possible. The following modal solution for displacements and stresses satisfies the simply supported boundary conditions:

$$\begin{aligned}
u &= U_{mn}\cos Mx\sin Ny\sin\Omega t; & \sigma_x &= S_{xmn}\sin Mx\sin Ny\sin\Omega t;\\
v &= V_{mn}\sin Mx\cos Ny\sin\Omega t; & \sigma_y &= S_{ymn}\sin MX\sin Ny\sin\Omega t;\\
w &= W_{mn}\sin Mx\sin Ny\sin\Omega t; & \sigma_z &= S_{zmn}\sin Mx\sin Ny\sin\Omega t;\\
\tau_{xz} &= T_{xzmn}\cos Mx\sin Ny\sin\Omega t; & \tau_{yz} &= T_{yzmn}\sin Mx\cos Ny\sin\Omega t;\\
\tau_{xy} &= T_{xymn}\cos Mx\cos Ny\sin\Omega t, & & \qquad (7)
\end{aligned}$$

where

$$M = m\pi/a \quad\text{and}\quad N = n\pi/b.$$

Here, a and b are the dimensions of the shell in the x- and y-direction; m and n are the number of half-waves in the x- and y-direction; and Ω is the radian frequency associated with the mode (m, n). In the above modal solution we note that U_{mn}, V_{mn} and W_{mn} are functions of the normal coordinate z and are to be determined as the solution of the following ODEs, which are obtained after substituting eqns (7) in eqns (6) as

$$L_{11}U_{mn}+L_{12}V_{mn}+L_{13}W_{mn}=0 \tag{8a}$$

$$L_{21}U_{mn}+L_{22}V_{mn}+L_{23}W_{mn}=0 \tag{8b}$$

$$L_{31}U_{mn}+L_{32}V_{mn}+L_{33}W_{mn}=0; \tag{8c}$$

where the differential operators L_{ij} are given by

$$\begin{aligned}
&L_{11}=a_1\frac{\mathrm{d}^2}{\mathrm{d}z^2}+a_2\frac{\mathrm{d}}{\mathrm{d}z}+a_3; && L_{12}=L_{21}=a_4\\
&L_{13}=a_5\frac{\mathrm{d}}{\mathrm{d}z}+a_6; && L_{22}=a_7\frac{\mathrm{d}^2}{\mathrm{d}z^2}+a_8\frac{\mathrm{d}}{\mathrm{d}z}+a_9\\
&L_{23}=a_{10}\frac{\mathrm{d}}{\mathrm{d}z}+a_{11}; && L_{31}=a_{12}\frac{\mathrm{d}}{\mathrm{d}z}+a_{13}\\
&L_{32}=a_{14}\frac{\mathrm{d}}{\mathrm{d}z}+a_{15}; && L_{33}=a_{16}\frac{\mathrm{d}^2}{\mathrm{d}z^2}+a_{17}\frac{\mathrm{d}}{\mathrm{d}z}+a_{18};
\end{aligned} \tag{9}$$

a_1–a_{18} appearing in eqns (9) have the following definitions:

$$\begin{aligned}
&a_1=C_{44};\quad a_2=\left[\frac{1}{R_1}+\frac{1}{R_2}\right]C_{44};\quad a_3=-\left[\frac{2}{R_1}+\frac{1}{R_2}\right]\frac{C_{44}}{R_1}-C_{11}M^2-C_{66}N^2+\rho\Omega^2\\
&a_4=-[C_{12}+C_{66}]MN;\quad a_5=[C_{23}+C_{44}]M;\quad a_6=\left[\frac{C_{11}+2C_{44}}{R_1}+\frac{C_{12}+C_{44}}{R_2}\right]M\\
&a_7=C_{55};\quad a_8=\left[\frac{1}{R_1}+\frac{1}{R_2}\right]C_{55};\quad a_9=-\left[\frac{1}{R_1}+\frac{2}{R_2}\right]\frac{C_{55}}{R_2}-C_{66}M^2-C_{22}N^2+\rho\Omega^2\\
&a_{10}=[C_{23}+C_{55}]N;\quad a_{11}=\left[\frac{C_{22}+2C_{55}}{R_2}+\frac{C_{12}+C_{55}}{R_1}\right]N;\quad a_{12}=-[C_{13}+C_{44}]M\\
&a_{13}=\left[\frac{C_{11}+C_{44}-C_{13}}{R_1}+\frac{C_{12}-C_{13}}{R_2}\right]M;\quad a_{14}=[C_{23}+C_{55}]N\\
&a_{15}=\left[\frac{C_{22}+C_{55}-C_{23}}{R_2}+\frac{C_{12}-C_{23}}{R_1}\right]N;\quad a_{16}=C_{33};\quad a_{17}=\left[\frac{1}{R_1}+\frac{1}{R_2}\right]C_{33}\\
&a_{18}=\left[\frac{C_{13}-C_{11}}{R_1^2}+\frac{C_{13}+C_{23}-2C_{12}}{R_1R_2}+\frac{C_{23}-C_{22}}{R_2^2}\right]-C_{44}M^2-C_{55}N^2+\rho\Omega^2.
\end{aligned} \tag{10}$$

The solution of eqns (8) are obtained by expressing U_{mn}, V_{mn} and W_{mn} in terms of a displacement potential function ϕ_{mn} as follows:

$$\begin{aligned}
U_{mn}&=(L_{12}L_{23}-L_{13}L_{22})\phi_{mn};\quad V_{mn}=(L_{13}L_{21}-L_{23}L_{11})\phi_{mn}\\
W_{mn}&=(L_{11}L_{22}-L_{12}L_{21})\phi_{mn}.
\end{aligned} \tag{11}$$

Substituting the above solution in eqns (8), it can be seen that eqns (8a) and (8b) are satisfied identically and eqn (8c) reduces to the following governing equation in ϕ_{mn}:

$$c_1\frac{\mathrm{d}^6\phi}{\mathrm{d}z^6}+c_2\frac{\mathrm{d}^5\phi}{\mathrm{d}z^5}+c_3\frac{\mathrm{d}^4\phi}{\mathrm{d}z^4}+c_4\frac{\mathrm{d}^3\phi}{\mathrm{d}z^3}+c_5\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2}+c_6\frac{\mathrm{d}\phi}{\mathrm{d}z}+c_7\phi=0. \tag{12}$$

In the above equation and in the subsequent analysis, subscripts "*mn*" have been dropped for the sake of simplicity and the expressions for c_1–c_7 are given in the Appendix. Since eqn (12) is of the order six, the solution ϕ involves six arbitrary constants and is sought in the form:

$$\phi = e^{\alpha z} \tag{13}$$

and six αs are obtained as the roots of the following equation:

$$c_1\alpha^6+c_2\alpha^5+c_3\alpha^4+c_4\alpha^3+c_5\alpha^2+c_6\alpha+c_7 = 0. \tag{14}$$

The exact expression for ϕ depends on the nature of the roots of eqn (14), for example, for six real and distinct roots we can write

$$\phi = A_1 e^{\alpha_1 z}+A_2 e^{\alpha_2 z}+A_3 e^{\alpha_3 z}+A_4 e^{\alpha_4 z}+A_5 e^{\alpha_5 z}+A_6 e^{\alpha_6 z}, \tag{15a}$$

or in the matrix notation as

$$\phi = \mathbf{F}\boldsymbol{\delta}; \tag{15b}$$

here $\boldsymbol{\delta}^t = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ is a column vector of six constants and $\mathbf{F} = \{F_1, F_2, F_3, F_4, F_5, F_6\}$ is a row vector in which F_1–F_6 are the coefficients of A_1–A_6, and are functions of the normal coordinate z. Here, A_1–A_6 are the six arbitrary constants to be determined using the following six traction-free surface conditions

$$\begin{aligned} &\tau_{xz} = 0; \quad \tau_{yz} = 0; \quad \sigma_z = 0 \quad (\text{at } z = +h/2) \\ &\tau_{xz} = 0; \quad \tau_{yz} = 0; \quad \sigma_z = 0 \quad (\text{at } z = -h/2). \end{aligned} \tag{16}$$

Once ϕ is obtained from eqn (15) the displacements U, V and W can be computed using eqns (11); strains can be computed from eqns (3); and the stresses from eqns (4) as

$$\begin{bmatrix} U \\ V \\ T_{xy} \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ b_9 & b_{10} & b_{11} & b_{12} \end{bmatrix} \begin{bmatrix} \phi^{\mathrm{III}} \\ \phi^{\mathrm{II}} \\ \phi^{\mathrm{I}} \\ \phi \end{bmatrix} \tag{17a}$$

$$\begin{bmatrix} W \\ T_{xz} \\ T_{yz} \end{bmatrix} = \begin{bmatrix} b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{18} & b_{19} & b_{20} & b_{21} & b_{22} \\ b_{23} & b_{24} & b_{25} & b_{26} & b_{27} \end{bmatrix} \begin{bmatrix} \sigma^{\mathrm{IV}} \\ \phi^{\mathrm{III}} \\ \sigma^{\mathrm{II}} \\ \phi^{\mathrm{I}} \\ \phi \end{bmatrix} \tag{17b}$$

$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = \begin{bmatrix} b_{28} & b_{29} & b_{30} & b_{31} & b_{32} & b_{33} \\ b_{34} & b_{35} & b_{36} & b_{37} & b_{38} & b_{39} \\ b_{40} & b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \end{bmatrix} \begin{bmatrix} \phi^{\mathrm{V}} \\ \phi^{\mathrm{IV}} \\ \phi^{\mathrm{III}} \\ \phi^{\mathrm{II}} \\ \phi^{\mathrm{I}} \\ \phi \end{bmatrix}. \tag{17c}$$

In the above, "I" indicates differentiation with respect to z and the expressions for b_1–b_{45} are given in the Appendix. Thus knowing the complete variation of the stresses and displacements in the z-direction, the frequency determinant can be set up by using the six, traction-free surface conditions (16) as

$$\begin{bmatrix} {}^{+}\mathbf{D}_{\sigma} \\ {}^{-}\mathbf{D}_{\sigma} \end{bmatrix} \boldsymbol{\delta} = \mathbf{D}\boldsymbol{\delta} = \mathbf{0}, \tag{18}$$

where ${}^{+}\mathbf{D}_{\sigma}$ and ${}^{-}\mathbf{D}_{\sigma}$ are (3×6) matrices defined as follows:

$${}^{+}\mathbf{D}_{\sigma} = \begin{bmatrix} b_{40} & b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ 0 & b_{18} & b_{19} & b_{20} & b_{21} & b_{22} \\ 0 & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{\mathrm{V}} \\ \mathbf{F}^{\mathrm{IV}} \\ \mathbf{F}^{\mathrm{III}} \\ \mathbf{F}^{\mathrm{II}} \\ \mathbf{F}^{\mathrm{I}} \\ \mathbf{F} \end{bmatrix}_{\text{at } z = h/2} . \tag{19a}$$

Also, we define the following matrix which will be useful in the analysis of laminated and sandwich-type shells

$${}^{+}\mathbf{D}_{u} = \begin{bmatrix} b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ 0 & b_{1} & b_{2} & b_{3} & b_{4} \\ 0 & b_{5} & b_{6} & b_{7} & b_{8} \end{bmatrix} \begin{bmatrix} \mathbf{F}^{\mathrm{IV}} \\ \mathbf{F}^{\mathrm{III}} \\ \mathbf{F}^{\mathrm{II}} \\ \mathbf{F}^{\mathrm{I}} \\ \mathbf{F} \end{bmatrix}_{\text{at } z = h/2} . \tag{19b}$$

Similarly, ${}^{-}\mathbf{D}_{\sigma}$ and ${}^{-}\mathbf{D}_{u}$ can be obtained by evaluating $\mathbf{F}$, $\mathbf{F}^{\mathrm{I}}$, etc. at $z = -h/2$, in eqns (19). It is to be noted here that the frequency (Ω) appears implicitly in the expressions of the coefficients of $\mathbf{D}$ and is to be determined by equating the determinant of $\mathbf{D}$ to zero. This completes the formulation of the problem and its solution for a shell with given geometric and material parameters (R_1, R_2, a, b, h and C_{ij}). It should be pointed out here that the above procedure holds for a single layered shell and in the case of laminated and sandwich shells, there are six arbitrary constants for each layer. The frequency determinant is to be set up by using the interface stress and displacement continuity conditions, in addition to the known traction-free surface conditions. For layers i and $i+1$ the interface conditions are:

$$\begin{bmatrix} U \\ z = -\frac{h}{2} \end{bmatrix}^{(i)} = \begin{bmatrix} U \\ z = +\frac{h}{2} \end{bmatrix}^{(i+1)} ; \quad \begin{bmatrix} V \\ z = -\frac{h}{2} \end{bmatrix}^{(i)} = \begin{bmatrix} V \\ z = +\frac{h}{2} \end{bmatrix}^{(i+1)}$$

$$\begin{bmatrix} W \\ z = -\frac{h}{2} \end{bmatrix}^{(i)} = \begin{bmatrix} W \\ z = +\frac{h}{2} \end{bmatrix}^{(i+1)} ; \quad \begin{bmatrix} T_{xz} \\ z = -\frac{h}{2} \end{bmatrix}^{(i)} = \begin{bmatrix} T_{xz} \\ z = +\frac{h}{2} \end{bmatrix}^{(i+1)}$$

$$\begin{bmatrix} T_{yz} \\ z = -\frac{h}{2} \end{bmatrix}^{(i)} = \begin{bmatrix} T_{yz} \\ z = +\frac{h}{2} \end{bmatrix}^{(i+1)} ; \quad \begin{bmatrix} S_{z} \\ z = -\frac{h}{2} \end{bmatrix}^{(i)} = \begin{bmatrix} S_{z} \\ z = +\frac{h}{2} \end{bmatrix}^{(i+1)} \tag{20a}$$

or in the matrix notation

$${}^{-}\mathbf{D}_{\sigma}^{(i)}\boldsymbol{\delta}^{(i)} = {}^{+}\mathbf{D}_{\sigma}^{(i+1)}\boldsymbol{\delta}^{(i+1)} \quad \text{and} \quad {}^{-}\mathbf{D}_{u}^{(i)}\boldsymbol{\delta}^{(i)} = {}^{+}\mathbf{D}_{u}^{(i+1)}\boldsymbol{\delta}^{(i+1)}. \tag{20b}$$

For example, if the shell is made up of two layers there will be 12 constants (A_1–A_{12}, six for each layer), to be determined using the six surface traction conditions [see eqns (16)] and six interface (three transverse stresses and three displacements) continuity conditions [see eqns (20)], which results in 12 homogeneous algebraic equations as

$$\begin{bmatrix} {}^{+}\mathbf{D}_{\sigma}^{(1)} & \mathbf{0} \\ {}^{-}\mathbf{D}_{\sigma}^{(1)} & -{}^{+}\mathbf{D}_{\sigma}^{(2)} \\ {}^{-}\mathbf{D}_{u}^{(1)} & -{}^{+}\mathbf{D}_{u}^{(2)} \\ \mathbf{0} & {}^{-}\mathbf{D}_{\sigma}^{(2)} \end{bmatrix} \begin{bmatrix} \delta^{(1)} \\ \delta^{(2)} \end{bmatrix} = \mathbf{D}\delta = \mathbf{0}. \tag{21}$$

It is to be noted here that when the ratios h/R_1 and h/R_2 are not small enough to be considered as $\ll 1$, the thickness of the shell is divided into a number of layers with smaller thickness values so that for any layer, the values of the ratios h/R_1 and h/R_2 become $\ll 1$; and the solution is obtained by using the interface stress and displacement continuity conditions (20) and the surface traction conditions (16). For laminated shells and sandwich-type shells for which h/R_1 and h/R_2 are not too small, each layer shall be divided into sub-layers with sufficiently smaller h/R_1 and h/R_2 ratios. However, a proper value for h/R_1 and h/R_2 can be chosen by conducting some numerical experiments and observing the convergence of the frequency values.

Furthermore, it is to be noted here that the corresponding equations for rectangular plates can be deduced from the present analysis by using $1/R_1 = 1/R_2 = 0$, and those for cylindrical shells by using $1/R_1 = 0$ and $R_2 = R$. Thus, the present procedure and the corresponding computer coding has been checked by computing the results for isotropic and orthotropic plates (Srinivas and Rao, 1970). Also, we note that the approximations made in eqns (2) hold true for rectangular plates and hence exact results can be obtained without subdividing the thickness of the plate. Since it is intended to compare the 2-D shell theories with the present exact 3-D analysis, a brief discussion of the 2-D theories has been included here for the sake of continuity and completeness.

TWO-DIMENSIONAL DOUBLE CURVED SHALLOW SHELL THEORY

The 2-D shell theories may be classified into three categories: the Thin Shell Theories (TST), the Shear Deformation Theories, and the Higher-Order Theories. The discussion of these various theories may be found in the publications by Bhimaraddi (1987) and Stein (1986). Here we discuss briefly the parabollic shear deformation theory (Bhimaraddi; 1984, 1987) in the present context of the doubly curved shells. In this theory, the displacements are expressed as

$$u = u_0 + f u_1 - z\frac{\partial w_0}{\partial x}; \quad v = v_0 + f v_1 - z\frac{\partial w_0}{\partial y}; \quad w = w_0 \tag{22}$$

where f and its first derivative (f^*) are functions of z only and are given as

$$f = z\left[1 - \frac{4z^2}{3h^2}\right]; \quad f^* = \frac{\mathrm{d}f}{\mathrm{d}z} = \left[1 - \frac{4z^2}{h^2}\right]. \tag{23}$$

In these equations u_0, v_0 and w_0 are the translations of a point on the shell middle surface; and u_1 and v_1 are the rotations of a point on the middle surface in addition to the usual flexural rotations $\partial w_0/\partial x$ and $\partial w_0/\partial y$. All the middle surface displacement parameters (viz. u_0, v_0, w_0, u_1, v_1) are functions of (x, y) only, and are independent of the z coordinate. The strains are written as

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + f\frac{\partial u_1}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + \frac{w_0}{R_1}; \quad \varepsilon_y = \frac{\partial v_0}{\partial y} + f\frac{\partial v_1}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} + \frac{w_0}{R_2}$$

$$\varepsilon_z = 0; \quad \gamma_{xz} = f^* u_1; \quad \gamma_{yz} = f^* v_1; \quad \gamma_{xy} = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + f\frac{\partial u_1}{\partial y} + f\frac{\partial v_1}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x\,\partial y}. \tag{24}$$

It may be observed that the transverse shear strains vanish on the top ($z = h/2$) and bottom ($z = -h/2$) surfaces of the shell and they vary parabolically across the thickness. It is to be noted here that the displacements corresponding to the Mindlin-type (in which the transverse shear strains are constant across the thickness) Constant Shear Deformation (CSD) theory, can be obtained if one uses $f = z$ and also the classical Thin Shell Theory can be deduced if one uses $f = 0$ in the above equations. In the case of CSD, the shear correction factor is to be used to correct the deficiencies in that theory (non-parabolic variation of shear stresses and non-vanishing of the shear stresses at the top and bottom surfaces of the shell). Further development of the theory follows a standard routine which can be obtained in Bhimaraddi (1984, 1987) and will not be repeated here for the sake of brevity.

DISCUSSION OF NUMERICAL RESULTS

It is to be said here that the present elasticity analysis yields an infinite number of frequencies for each combination of (m, n) values, whereas, the PSD and CSD yield five frequencies and TST yields three frequencies. Of these frequencies, the lowest frequency corresponds to the flexural mode of vibration and only this frequency has been computed and discussed in this study. In the numerical computations of CSD, the shear correction factor used corresponds to $\pi^2/12$. Since this is the first time that the analysis for shallow (on rectangular planform) cylindrical shells and spherical shells has been given using the 3-D equations, all the numerical results have been given in a tabular form which are intended to serve as bench-mark values for future studies. The typical orthotropic material properties correspond to:

$$\frac{E_x}{E_y} = 25; \quad \frac{E_z}{E_y} = 1; \quad \frac{G_{xz}}{E_y} = \frac{G_{xy}}{E_y} = \frac{1}{2}; \quad \frac{G_{yz}}{E_y} = \frac{1}{5}; \quad \mu_{xy} = 0.25; \quad \mu_{zx} = 0.03; \quad \mu_{yz} = 0.4.$$

Here E_x, E_y and E_z are the Young's moduli; G_{xy}, G_{xz} and G_{yz} are the shear moduli; and μ_{xy}, μ_{zx} and μ_{yz} are the Poisson ratios. As noted earlier in the case of thick shells for which the assumptions in eqns (2) are not valid, the shell thickness has to be divided into a number of layers such that for each layer, eqns (2) hold true. A proper value for h/R can be fixed by conducting some numerical experiments in which the convergence of the frequency can be observed.

Tables 1, 2 and 3 show the convergence studies for homogeneous spherical shells on rectangular planform with different h/a and R/a ratios (Table 1), and with different wave numbers (Table 2), and with different aspect ratios (Table 3). It may be observed from these tables that as the number of divisions in the shell thickness increases, the frequency values converge monotonically from above. This pattern of convergence is completely in agreement with Rayleigh's principle and this also validates the present approach. Furthermore, this suggests that the frequencies from the present analysis (without dividing the shell thickness) are the upper bounds to the exact values. It may be seen from Table 1 that as h/a ratio of the shell increases and as R/a ratio decreases, one has to divide the thickness of the shell into a greater number of layers to achieve the convergence of the results. For a

Table 1. Convergence studies on an orthotropic spherical shell frequency ($\bar{\Omega}$) with different h/a and R/a ratios ($\bar{\Omega} = \Omega a\sqrt{\rho/E_y}$; $a/b = 1$) ($E_x = 25E_y$; $G_{xz} = G_{xy} = \frac{1}{2}E_y$; $G_{yz} = \frac{1}{5}E_y$; $\mu_{xy} = \frac{1}{4}$; $\mu_{zx} = 0.03$; $\mu_{yz} = 0.4$)

	$m = n = 1$, $a/b = 1$, $R/a = 1$			$m = n = 1$, $a/b = 1$, $h/a = 0.1$		
N_d†	$h/a = 0.05$	$h/a = 0.1$	$h/a = 0.15$	$R/a = 3$	$R/a = 5$	$R/a = 10$
1	1.24313	1.50152	1.70293	1.27975	1.25902	1.25011
2	1.22140	1.44837	1.63018	1.27070	1.25565	1.24926
5	1.21511	1.43177	1.60515	1.26799	1.25464	1.24900
10	1.21421	1.42933	1.60137	1.26760	1.25450	1.24896
15	1.21404	1.42888	1.60066	1.26753	1.25447	1.24896
20	1.21403	1.42872	1.60041	1.26750	1.25446	1.24896

† Number of divisions in the shell thickness.

Table 2. Convergence studies on an orthotropic spherical shell frequency ($\bar{\Omega}$) with different mode numbers $(m = n)$ $(\bar{\Omega} = \Omega a\sqrt{\rho/E_y}$; $a/b = 1$; $R/a = 1$; $h/a = 0.1)$ $(E_x = 25E_y$; $G_{xz} = G_{xy} = \frac{1}{2}E_y$; $G_{yz} = \frac{1}{5}E_y$; $\mu_{xy} = \frac{1}{4}$; $\mu_{zx} = 0.03$; $\mu_{yz} = 0.4)$

N_d	$m = n = 2$	$m = n = 3$	$m = n = 4$	$m = n = 5$	$m = n = 6$	$m = n = 7$
1	3.61086	6.09921	8.66825	11.27115	13.89055	16.51733
2	3.55698	6.05942	8.63790	11.24693	13.87052	16.50033
5	3.53794	6.04322	8.62408	11.23496	13.86002	16.49099
10	3.53503	6.04059	8.62166	11.23269	13.85787	16.48895
15	3.53449	6.04009	8.62119	11.23225	13.85744	16.48853
20	3.53429	6.03992	8.62103	11.23209	13.85729	16.48838

Table 3. Convergence studies on an orthotropic translational shell frequency ($\bar{\Omega}$) with different aspect (a/b) ratios $(\bar{\Omega} = \Omega a\sqrt{\rho/E_y}$; $m = n = 1$; $R/a = 1$; $h/a = 0.1)$ $(E_x = 25E_y$; $G_{xz} = G_{xy} = \frac{1}{2}E_y$; $G_{yz} = \frac{1}{5}E_y$; $\mu_{xy} = \frac{1}{4}$; $\mu_{zx} = 0.03$; $\mu_{yz} = 0.4)$

N_d	$a/b = 1.5$	$a/b = 2$	$a/b = 2.5$	$a/b = 3$	$a/b = 3.5$	$a/b = 5$
1	1.76559	2.13501	2.59033	3.11262	3.68538	5.58788
2	1.71716	2.09170	2.55159	3.07769	3.65358	5.56270
5	1.70218	2.07842	2.53980	3.06709	3.64392	5.55487
10	1.69998	2.07648	2.53808	3.06555	3.64252	5.55374
15	1.69957	2.07612	2.53776	3.06526	3.64225	5.55353
20	1.69943	2.07600	2.53765	3.06516	3.64216	5.55345

Table 4. Comparison of fundamental frequencies ($\bar{\Omega}$) for orthotropic homogeneous and two-layered (0/90) cylindrical shells for different R/a and h/a ratios $(a/b = 1$, $1/R_1 = 0$, $\bar{\Omega} = \Omega a\sqrt{\rho/E_y})$

		Homogeneous cylinder			0/90 Cylinder		
R/a		$h/a = 0.05$	$h/a = 0.1$	$h/a = 0.15$	$h/a = 0.05$	$h/a = 0.1$	$h/a = 0.15$
1	3-D	0.89171	1.32416	1.61690	0.78683	1.04085	1.29099
	PSD	0.89791	1.33745	1.63718	0.79993	1.09189	1.38174
	CSD	0.89124	1.29858	1.55779	0.79798	1.07475	1.33274
	TST	0.93015	1.57257	2.23906	0.80580	1.14313	1.54124
2	3-D	0.76632	1.26744	1.59247	0.57252	0.93627	1.25377
	PSD	0.76857	1.27076	1.59664	0.58000	0.95664	1.28933
	CSD	0.76045	1.22790	1.51092	0.57733	0.93653	1.23527
	TST	0.80747	1.52693	2.24197	0.58723	1.01398	1.45781
3	3-D	0.73968	1.25625	1.58789	0.52073	0.91442	1.24500
	PSD	0.74095	1.25736	1.58856	0.52516	0.92642	1.90563
	CSD	0.73246	1.21368	1.50158	0.52222	0.90563	1.21316
	TST	0.78151	1.51784	2.24256	0.53294	0.98505	1.43751
4	3-D	0.73004	1.25227	1.58529	0.50110	0.90613	1.24090
	PSD	0.73094	1.25259	1.58569	0.50415	0.91506	1.25977
	CSD	0.72231	1.20860	1.49826	0.50109	0.89403	1.20454
	TST	0.77213	1.51461	2.24277	0.51217	0.97408	1.42910
5	3-D	0.72552	1.25033	1.58424	0.49167	0.90200	1.23849
	PSD	0.72625	1.25036	1.58436	0.49402	0.90953	1.25551
	CSD	0.71755	1.20624	1.49671	0.49091	0.88840	1.20020
	TST	0.76773	1.51310	2.24286	0.50216	0.96870	1.42464
10	3-D	0.71944	1.24735	1.58254	0.47859	0.89564	1.23374
	PSD	0.71992	1.24738	1.58257	0.47997	0.90150	1.24875
	CSD	0.71114	1.20307	1.49464	0.47677	0.88026	1.19342
	TST	0.76182	1.51108	2.24299	0.48827	0.96074	1.41709
20	3-D	0.71791	1.24633	1.58210	0.47509	0.89341	1.23140
	PSD	0.71833	1.24663	1.58212	0.47625	0.89904	1.24626
	CSD	0.70952	1.20227	1.49412	0.47304	0.87779	1.19100
	TST	0.76033	1.51058	2.24303	0.48459	0.95819	1.41400
∞	3-D	0.71739	1.24612	1.58121	0.47365	0.89179	1.22905
	PSD	0.71780	1.24638	1.58197	0.47483	0.89761	1.24437
	CSD	0.70898	1.20201	1.49394	0.47161	0.87640	1.18923
	TST	0.75983	1.51041	2.24304	0.48317	0.95661	1.41139

3-D—Present 3-D analysis; PSD—Parabolic Shear Deformation theory; CSD—Constant Shear Deformation theory; TST—Thin Shell Theory.

shell with $h/R = 0.01$ ($h/a = 0.1$ and $R/a = 10$), almost exact results are obtained with $N_d = 1$ as may be inferred from this table. From Tables 2 and 3 one may observe that a greater number of divisions have to be made in the shell thickness for lower modes and lower aspect ratios to achieve the required convergence. Thus one can conclude that if the h/R ratio of the shell is $\leqslant 0.01$, no division of the thickness of the shell is required. It may also be noted that for layered shells with $h/R \leqslant 0.01$ at its middle surface, no sub-division of each layer is necessary. Thus all the numerical results presented in this paper have been obtained by keeping the h/R ratio of <0.01 for any sub-layer.

Tables 4–6 depict frequency values for homogeneous and two-layered cylindrical shells with different h/a, R/a and (m, n) values, and those for spherical shells are shown in Tables 7–9. One may observe from these tables that, in most cases, the frequency values for homogeneous shells are higher as compared to those of the two-layered (0/90) shells. But for higher modes, two-layered shells give higher frequencies than the homogeneous shells. It may be seen from these tables that PSD and TST predict consistently higher frequency values when compared with the 3-D analysis, whereas CSD predicts lower values in most cases but for some ($R/a = 1$) two-layered shells, it predicts higher values.

From Tables 4 and 7 it may be said here that the errors in the 2-D theories increase with increasing shell thickness. The errors in PSD and CSD are negligible for cylindrical shells with $h/a = 0.05$ whereas, even for such a thin cylindrical shell the error in TST is about 4.3% for homogeneous ($R/a = 1$) shells, and this error increases to 6% for homogeneous plates. Also, it may be said here that the errors in the 2-D theories are higher for spherical shells as compared to those for cylindrical shells. For thick shells, the frequency values from TST are in greater discrepancy when compared with the 3-D analysis, and

Table 5. Comparison of fundamental frequencies ($\bar{\Omega}$) for orthotropic homogeneous cylindrical shells for different wave numbers ($a/b = 1$, $R_2/a = 1$, $h/a = 0.1$, $1/R_1 = 0$, $\bar{\Omega} = \Omega a\sqrt{\rho/E_r}$)

n		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	3-D	1.32416	3.42856	5.65598	7.88524	10.10983	12.31165
	PSD	1.33745	3.43487	5.68570	8.00360	10.41998	12.96862
	CSD	1.29858	3.19538	5.13383	7.03869	8.91857	10.78286
	TST	1.57257	5.26048	7.34788	9.41284	11.53382	13.68637
2	3-D	1.65929	3.59532	5.77045	7.09557	10.18638	12.39924
	PSD	1.69031	3.60790	5.80137	8.09173	10.49072	13.02658
	CSD	1.63768	3.36074	5.24850	7.13296	9.00273	10.86098
	TST	1.99961	5.85093	9.08880	10.84096	12.72805	14.70735
3	3-D	2.50573	4.08989	6.10449	8.22842	10.38124	12.57257
	PSD	2.53918	4.10523	6.13436	8.34022	10.68794	13.18882
	CSD	2.44409	3.83876	5.57354	7.38087	9.20609	11.03596
	TST	3.09808	6.45360	11.36729	12.87246	14.50117	16.26705
4	3-D	3.64582	4.89994	6.68608	8.67551	10.75359	12.87744
	PSD	3.67091	4.91181	6.71169	8.78055	11.04046	13.48058
	CSD	3.49515	4.59461	6.12261	7.80606	9.55271	11.32941
	TST	4.77389	7.53134	13.18665	15.26604	16.66900	18.22746
5	3-D	4.92951	5.93701	7.48804	9.31267	11.27750	13.32086
	PSD	4.94282	5.94121	7.50705	9.40951	11.55338	13.90986
	CSD	4.65406	5.53630	6.85888	8.39601	10.04962	11.74416
	TST	6.89750	9.13471	14.23784	17.87185	19.09700	20.47352
6	3-D	6.28273	7.11414	8.45781	10.11238	11.94887	13.89574
	PSD	6.28481	7.11024	8.47035	10.20163	12.21433	14.47086
	CSD	5.85751	6.58546	7.73038	9.12068	10.65287	12.27984
	TST	9.36898	11.20323	15.62888	20.59374	21.59722	22.92106
7	3-D	7.66909	8.37330	9.54541	11.04051	12.74557	14.58772
	PSD	7.66367	8.36430	9.55452	11.12540	13.00356	15.15128
	CSD	7.07696	7.69351	8.69288	9.84800	11.36766	12.89485
	TST	12.10934	13.64076	17.42526	23.26989	24.41370	25.51177
8	3-D	9.07057	9.68036	10.03301	12.06614	13.64472	15.38031
	PSD	9.06401	9.67213	10.72479	12.15254	13.90077	15.93704
	CSD	8.29955	8.83331	9.71513	10.85103	12.16411	13.60064
	TST	15.05423	16.35665	19.58430	25.20054	27.21024	28.20581

Table 6. Comparison of fundamental frequencies ($\bar{\Omega}$) for orthotropic two-layered (0/90) cylindrical shells for different wave numbers ($a/b = 1$, $R_2/a = 1$, $h/a = 0.1$, $1/R_1 = 0$, $\bar{\Omega} = \Omega a\sqrt{\rho/E_r}$)

n		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	3-D	1.04085	2.41276	4.11579	5.93372	7.78184	9.62817
	PSD	1.09189	2.45460	4.23881	6.21438	8.29333	10.45184
	CSD	1.07475	2.34178	3.89006	5.51004	7.13494	8.74763
	TST	1.14313	2.83420	5.57063	9.18375	13.46425	17.41943
2	3-D	2.09560	3.00690	4.47600	6.17789	7.96114	9.76724
	PSD	2.23166	3.11351	4.63052	6.47026	8.47328	10.58518
	CSD	2.10261	2.91833	4.21882	5.71108	7.26438	8.83414
	TST	2.64497	3.77786	6.19582	9.64978	13.84158	18.53581
3	3-D	3.79493	4.40105	5.53384	6.99587	8.61936	10.31605
	PSD	4.03139	4.60653	5.75652	7.32258	9.14262	11.12949
	CSD	3.64408	4.18708	5.18111	6.45052	7.85330	9.31876
	TST	5.46968	6.17649	7.94639	10.85995	14.66478	19.07848
4	3-D	5.63314	6.08162	6.96436	8.18815	9.62322	11.17793
	PSD	6.03517	6.45147	7.32643	8.61674	10.21128	12.02542
	CSD	5.27341	5.67727	6.44935	7.50530	8.73691	10.07120
	TST	9.28491	9.64745	10.85959	13.08162	18.26925	20.20189
5	3-D	7.48761	7.85500	8.57043	9.60352	10.86447	12.27431
	PSD	8.13620	8.46125	9.15315	10.21892	11.59781	13.22603
	CSD	6.90648	7.23117	7.85529	8.74126	9.81579	11.01692
	TST	13.49825	13.84378	14.66423	16.26830	18.76569	22.07201
6	3-D	8.68429	9.49798	10.25407	11.14312	12.25850	13.53670
	PSD	10.31101	10.57769	11.14483	12.07983	11.02184	12.10090
	CSD	8.52564	8.80021	9.32326	10.07983	11.02184	12.10090
	TST	17.17463	18.48576	19.06173	20.19360	22.06780	24.71657
7	3-D	11.18770	11.44636	11.85606	12.74791	13.74617	14.91182
	PSD	12.56186	12.78788	13.26611	14.02998	15.07075	16.36456
	CSD	10.12930	10.36944	10.82037	11.47820	12.31147	13.28365
	TST	19.13495	23.24020	23.80560	24.61440	25.99090	28.04362
8	3-D	12.99253	13.22186	13.68342	14.29142	15.28707	16.35888
	PSD	14.98969	15.09425	15.50632	16.16905	17.08463	18.24131
	CSD	11.71966	11.83443	12.33183	12.91299	13.65742	14.53780
	TST	21.01155	27.22880	28.67141	29.32695	30.34223	31.90483

Table 7. Comparison of fundamental frequencies ($\bar{\Omega}$) for orthotropic homogeneous and two-layered (0/90) spherical shells for different R/a and h/a ratios ($a/b = 1$, $\bar{\Omega} = \Omega a\sqrt{\rho/E_T}$)

R/a		Homogeneous shell			0/90 Shell		
		$h/a = 0.05$	$h/a = 0.1$	$h/a = 0.15$	$h/a = 0.05$	$h/a = 0.1$	$h/a = 0.15$
1	3-D	1.21403	1.42872	1.60041	1.29835	1.39974	1.51936
	PSD	1.28525	1.60538	1.84983	1.32595	1.49075	1.68141
	CSD	1.28089	1.57496	1.78352	1.32483	1.48008	1.64797
	TST	1.30657	1.79568	2.37996	1.33000	1.52391	1.78940
2	3-D	0.87702	1.29295	1.58068	0.79577	1.05528	1.31111
	PSD	0.90697	1.35302	1.65888	0.81059	1.09708	1.38083
	CSD	0.90021	1.31347	1.57769	0.80870	1.08054	1.33375
	TST	0.93961	1.59277	2.28118	0.81618	1.14507	1.52705
3	3-D	0.79315	1.26750	1.58194	0.64044	0.96917	1.26650
	PSD	0.80865	1.29559	1.61714	0.64949	0.99330	1.30815
	CSD	0.80093	1.25354	1.53231	0.64713	0.97455	1.25698
	TST	0.84569	1.54813	2.26036	0.65602	1.04657	1.46512
4	3-D	0.76111	1.25859	1.58279	0.57419	0.93637	1.25032
	PSD	0.77049	1.27445	1.60196	0.58038	0.95306	1.28092
	CSD	0.76234	1.23141	1.51576	0.57775	0.93332	1.22810
	TST	0.80950	1.53186	2.25285	0.58749	1.00862	1.44211
5	3-D	0.74572	1.25446	1.58326	0.54039	0.92065	1.24272
	PSD	0.75203	1.26446	1.59482	0.54500	0.93361	1.26797
	CSD	0.74366	1.22096	1.50798	0.54219	0.91338	1.21434
	TST	0.79206	1.52420	2.24934	0.55247	0.99034	1.43120
10	3-D	0.72460	1.24896	1.58396	0.49127	0.89912	1.23249
	PSD	0.72654	1.25094	1.58520	0.49341	0.90679	1.25034
	CSD	0.71784	1.20679	1.49748	0.49031	0.88584	1.19559
	TST	0.76804	1.51388	2.24462	0.50149	0.96519	1.41639
20	3-D	0.71920	1.24751	1.58415	0.47812	0.89363	1.22992
	PSD	0.72000	1.24752	1.58278	0.47955	0.89992	1.24586
	CSD	0.71121	1.20321	1.49483	0.47636	0.87877	1.19083
	TST	0.76189	1.51128	2.24343	0.48782	0.95876	1.41264
∞	3-D	0.71739	1.24612	1.58121	0.47365	0.89179	1.22905
	PSD	0.71780	1.24638	1.58197	0.47483	0.89761	1.24437
	CSD	0.70898	1.20201	1.49394	0.47161	0.87640	1.18923
	TST	0.75983	1.51041	2.24304	0.48317	0.95661	1.41139

Table 8. Comparison of fundamental frequencies ($\bar{\Omega}$) for orthotropic homogeneous spherical shells for different wave numbers ($a/b = 1$, $R_1/a = R_2/a = 1$, $h/a = 0.1$, $\bar{\Omega} = \Omega a\sqrt{\rho/E_y}$)

n		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	3-D	1.42872	3.27331	5.50649	7.75736	9.99933	12.23533
	PSD	1.60538	3.46843	5.68772	7.99846	10.41384	12.96535
	CSD	1.57496	3.23244	5.13567	7.03161	8.90895	10.77278
	TST	1.78568	5.36828	7.35255	9.41377	11.53411	13.68648
2	3-D	2.07600	3.53429	5.66760	7.87736	10.09809	12.32027
	PSD	2.21695	3.71369	5.83763	8.10798	10.49975	13.03416
	CSD	2.18070	3.47708	5.28908	7.14946	9.00877	13.86177
	TST	2.44004	5.91644	9.10874	10.84443	12.72913	14.70778
3	3-D	3.06516	4.10922	6.03992	8.15349	10.31955	12.50657
	PSD	3.17490	4.26364	6.19484	8.37051	10.70610	13.20203
	CSD	3.10694	4.01268	5.64279	7.41557	9.22524	11.04674
	TST	3.59212	6.52707	11.44004	12.87985	14.50339	16.26794
4	3-D	4.32251	4.98212	6.65598	8.62103	10.59572	12.82174
	PSD	4.34029	5.10655	6.79153	8.82192	11.06554	13.49799
	CSD	4.20587	4.80924	6.21447	7.85502	9.58212	11.34823
	TST	5.22100	7.61657	13.29738	15.27933	16.67263	18.22890
5	3-D	5.55345	6.06453	7.48802	9.27688	11.23209	13.27414
	PSD	5.61708	6.16032	7.60257	9.46037	11.58443	13.93101
	CSD	5.38348	5.78017	6.96917	8.45684	10.07850	11.76950
	TST	7.26633	9.22016	14.27403	17.89553	19.10233	20.47558
6	3-D	6.90523	7.27443	8.48318	10.09355	11.91499	13.85729
	PSD	6.95129	7.34651	8.57864	10.26078	12.25076	14.49551
	CSD	6.59072	6.85176	7.85617	9.18182	10.69807	12.30179
	TST	9.65830	11.27930	15.65315	20.64238	21.70469	22.92382
7	3-D	8.28209	8.55810	9.59200	11.03685	12.72238	14.55700
	PSD	8.31543	8.61333	9.67339	11.19186	13.04496	15.17923
	CSD	7.80437	7.97794	8.83210	10.02834	11.41942	12.93077
	TST	12.32775	13.70261	17.44443	23.43373	24.42396	25.51531
8	3-D	9.66931	9.88399	10.77771	12.07596	13.63132	15.35680
	PSD	9.69664	9.93085	10.85258	12.22547	13.94671	15.96812
	CSD	9.01444	9.13267	9.86628	10.93976	12.22191	13.64107
	TST	15.21175	16.40283	19.59905	25.27066	27.22444	28.21021

Table 9. Comparison of fundamental frequencies ($\bar{\Omega}$) for orthotropic two-layered (0/90) spherical shells for different wave numbers ($a/b = 1$, $R_1/a = R_2/a = 1$, $h/a = 0.1$, $\bar{\Omega} = \Omega a\sqrt{\rho/E_y}$)

n		$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	3-D	1.39974	2.43873	4.05310	5.84554	7.68958	8.79732
	PSD	1.49075	2.56254	4.25854	6.20360	8.27291	10.43078
	CSD	1.48008	2.46151	3.92396	5.51209	7.12350	8.73011
	TST	1.52391	2.89707	5.50027	9.02660	13.24407	17.41747
2	3-D	2.44203	3.04522	4.41687	5.79381	8.03207	9.68411
	PSD	2.62378	3.23980	4.60005	6.46350	8.54558	10.56502
	CSD	2.52929	3.06519	4.26788	5.72230	7.25997	8.82298
	TST	2.92179	3.82427	6.11914	9.48366	13.60235	18.23648
3	3-D	4.08410	4.43275	5.47411	6.90512	8.52349	10.22652
	PSD	4.36988	4.72430	5.78704	7.31569	9.12123	11.10530
	CSD	4.04792	4.33741	5.23700	6.46472	7.84913	9.30665
	TST	5.54551	6.13159	7.83020	10.67702	14.42092	18.78285
4	3-D	5.71021	6.11283	6.90916	8.09839	9.52445	11.08302
	PSD	6.34133	6.56740	7.36517	8.61625	10.19310	12.00174
	CSD	5.66932	5.83753	6.51731	7.52796	8.73729	10.06090
	TST	9.03503	9.49056	10.67910	12.86521	16.01181	19.90538
5	3-D	7.40020	7.89044	8.52272	9.51883	10.76682	12.17710
	PSD	8.41897	8.57892	9.20134	10.22867	11.58767	13.20768
	CSD	7.29670	7.40360	7.93734	8.77559	9.82431	11.01175
	TST	12.97093	13.55461	14.40547	16.00500	18.48406	21.76722
6	3-D	9.03246	9.69485	10.21499	11.06553	12.16473	13.44009
	PSD	10.57303	10.69708	11.20196	12.05996	13.23292	14.67443
	CSD	8.90756	8.98359	9.41883	10.12649	11.04010	12.10281
	TST	18.43406	18.01580	18.71242	19.87494	21.75386	24.39637
7	3-D	11.43566	11.49662	11.93697	12.67820	13.65780	14.81747
	PSD	12.80345	12.90734	13.33031	14.06064	15.08097	16.36371
	CSD	8.90756	8.98359	9.41883	10.12649	11.04010	12.10281
	TST	16.43406	18.01580	18.71242	19.87494	21.75386	24.39637
8	3-D	13.24313	13.28095	13.66228	14.28615	15.20492	16.26823
	PSD	15.11949	15.21162	15.57526	16.20775	17.10415	18.24960
	CSD	12.07631	12.13174	12.44884	12.98151	13.69501	14.55550
	TST	20.85831	26.32647	28.07046	28.87560	29.94918	31.53944

hence make TST unacceptable for thick shells. As the radius of the shell increases, the error in PSD decreases and that in CSD increases. It may be seen from these tables that the predictions of PSD are remarkably accurate, even for thick shells with higher (m, n) values than those of CSD, when compared with the 3-D analysis.

CONCLUSIONS

The three-dimensional elasticity solution for free vibration of doubly curved, shallow shells on rectangular planform and made of an orthotropic material has been presented. Using the assumption that the thickness to radius ratio is negligible compared to unity, the governing equilibrium equations have been reduced to differential equations with constant coefficients. Furthermore, by dividing the shell thickness into a number of layers, such that their individual thickness to radius ratio is kept as low as practicable (and in this study it is shown to be 1/100), very accurate results were obtained for thick shallow shells. Numerical results indicate that the parabolic shear deformation theory and the thin shell theory consistency overestimate the frequencies, whereas the Mindlin-type constant shear deformation theory underestimates the frequencies in most cases, except in some cases when compared with the present 3-D analysis. This indicates that the frequencies from the parabolic shear deformation theory and the classical shell theory are the upper bounds (bounds being narrowed in the case of the parabolic shear deformation theory), whereas those from the Mindlin-type constant shear deformation theory are the lower bounds to the actual values from the 3-D analysis. Comparison studies also indicate that the thin shell theory results are unacceptable for thick shells with a thickness-to-radius ratio of more than 1/20. The present analysis can easily be extended to shallow shell surfaces with twist, such as a hyperbolic paraboloid (hypar shell), by including the twist term in the strain-displacement relations (1) and the equilibrium equations (5).

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APPENDIX

Definitions of c_1–c_7 appearing in eqns (12) and (14):

$$c_1 = a_{16}b_{13};\quad c_2 = a_{16}b_{14}+a_{17}b_{13}$$
$$c_3 = a_{16}b_{15}+a_{17}b_{14}+a_{18}b_{13}+a_{12}b_1+a_{14}b_5$$
$$c_4 = a_{16}b_{16}+a_{17}b_{15}+a_{18}b_{14}+a_{12}b_2+a_{13}b_1+a_{14}b_6+a_{15}b_5$$
$$c_5 = a_{16}b_{17}+a_{17}b_{16}+a_{18}b_{15}+a_{12}b_3+a_{13}b_2+a_{14}b_7+a_{15}b_6$$
$$c_6 = a_{17}b_{17}+a_{18}b_{16}+a_{12}b_4+a_{13}b_3+a_{14}b_8+a_{15}b_7$$
$$c_7 = a_{18}b_{17}+a_{13}b_4+a_{15}b_8.$$

Definitions for b_1–b_{45} appearing in eqn (17):

$$b_1 = -a_5a_7;\quad b_2 = -a_5a_8-a_6a_7;\quad b_3 = a_4a_{10}-a_5a_9-a_6a_8$$
$$b_4 = a_4a_{11}-a_6a_9;\quad b_5 = -a_1a_{10};\quad b_6 = -a_1a_{11}-a_2a_{10}$$
$$b_7 = -a_3a_{10}-a_2a_{11}+a_4a_5;\quad b_8 = -a_3a_{11}+a_4a_6$$
$$b_9 = (Nb_1+Mb_5)C_{66};\quad b_{10} = (Nb_2+Mb_6)C_{66};\quad b_{11} = (Nb_3+Mb_7)C_{66}$$
$$b_{12} = (Nb_4+Mb_8)C_{66};\quad b_{13} = a_1a_7;\quad b_{14} = a_1a_8+a_2a_7$$
$$b_{15} = a_1a_9+a_2a_8+a_3a_7;\quad b_{16} = a_2a_9+a_3a_8;\quad b_{17} = a_3a_9-a_4^2$$
$$b_{18} = (b_1+Mb_{13})C_{44};\quad b_{19} = (b_2+Mb_{14}-b_1/R_1)C_{44};\quad b_{20} = (b_3+Mb_{15}-b_2/R_1)C_{44}$$
$$b_{21} = (b_4+Mb_{16}-b_3/R_1)C_{44};\quad b_{22} = (Mb_{17}-b_4/R_1)C_{44};\quad b_{23} = (b_5+Nb_{13})C_{55}$$
$$b_{24} = (b_6+Nb_{14}-b_5/R_2)C_{55};\quad b_{25} = (b_7+Nb_{15}-b_6/R_2)C_{55}$$
$$b_{26} = (b_8+Nb_{16}-b_7/R_2)C_{55};\quad b_{27} = (Nb_{17}-b_8/R_2)C_{55};\quad b_{28} = b_{13}C_{13}$$
$$b_{29} = \left[\frac{C_{11}}{R_1}+\frac{C_{12}}{R_2}\right]b_{13}+C_{13}b_{14};\quad b_{30} = \left[\frac{C_{11}}{R_1}+\frac{C_{12}}{R_2}\right]b_{14}+C_{13}b_{15}-C_{11}Mb_1-C_{12}Nb_5$$
$$b_{31} = \left[\frac{C_{11}}{R_1}+\frac{C_{12}}{R_2}\right]b_{15}+C_{13}b_{16}-C_{11}Mb_2-C_{12}Nb_6$$
$$b_{32} = \left[\frac{C_{11}}{R_1}+\frac{C_{12}}{R_2}\right]b_{16}+C_{13}b_{17}-C_{11}Mb_3-C_{12}Nb_7$$
$$b_{33} = \left[\frac{C_{11}}{R_1}+\frac{C_{12}}{R_2}\right]b_{17}-C_{11}Mb_4-C_{12}Nb_8;\quad b_{34} = C_{23}b_{13}$$
$$b_{35} = \left[\frac{C_{12}}{R_1}+\frac{C_{22}}{R_2}\right]b_{13}+C_{23}b_{14};\quad b_{36} = \left[\frac{C_{12}}{R_1}+\frac{C_{22}}{R_2}\right]b_{14}+C_{23}b_{15}-C_{12}Mb_1-C_{22}Nb_5$$
$$b_{37} = \left[\frac{C_{12}}{R_1}+\frac{C_{22}}{R_2}\right]b_{15}+C_{23}b_{16}-C_{12}Mb_2-C_{22}Nb_6$$
$$b_{38} = \left[\frac{C_{12}}{R_1}+\frac{C_{22}}{R_2}\right]b_{16}+C_{23}b_{17}-C_{12}Mb_3-C_{22}Nb_7$$
$$b_{39} = \left[\frac{C_{12}}{R_1}+\frac{C_{22}}{R_2}\right]b_{17}-C_{12}Mb_4-C_{22}Nb_8;\quad b_{40} = C_{33}b_{13}$$
$$b_{41} = \left[\frac{C_{13}}{R_1}+\frac{C_{23}}{R_2}\right]b_{13}+C_{33}b_{14};\quad b_{42} = \left[\frac{C_{13}}{R_1}+\frac{C_{23}}{R_2}\right]b_{14}+C_{33}b_{15}-C_{13}Mb_1-C_{23}Nb_5$$
$$b_{43} = \left[\frac{C_{13}}{R_1}+\frac{C_{23}}{R_2}\right]b_{15}+C_{33}b_{16}-C_{13}Mb_2-C_{23}Nb_6$$
$$b_{44} = \left[\frac{C_{13}}{R_1}+\frac{C_{23}}{R_2}\right]b_{16}+C_{33}b_{17}-C_{13}Mb_3-C_{23}Nb_7$$
$$b_{45} = \left[\frac{C_{13}}{R_1}+\frac{C_{23}}{R_2}\right]b_{17}-C_{13}Mb_4-C_{23}Nb_8.$$